

A-level  
**FURTHER MATHEMATICS**  
**7367/2**

Paper 2

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Mark scheme

June 2021

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Version: 1.0 Final Mark Scheme



Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

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## Mark scheme instructions to examiners

### General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

### Key to mark types

|   |   |
|---|---|
| M | mark is for method  |
| R | mark is for reasoning   |
| A | mark is dependent on M marks and is for accuracy              |
| B | mark is independent of M marks and is for method and accuracy |
| E | mark is for explanation                                       |
| F | follow through from previous incorrect result                 |

### Key to mark scheme abbreviations

|         |   |
|---------|---|
| CAO     | correct answer only   |
| CSO     | correct solution only   |
| ft      | follow through from previous incorrect result                     |
| 'their' | indicates that credit can be given from previous incorrect result |
| AWFW    | anything which falls within                                       |
| AWRT    | anything which rounds to  |
| ACF     | any correct form  |
| AG      | answer given  |
| SC      | special case  |
| OE      | or equivalent   |
| NMS     | no method shown   |
| PI      | possibly implied  |
| sf      | significant figure(s)   |
| dp      | decimal place(s)  |

Examiners should consistently apply the following general marking principles

### **No Method Shown**

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

**Otherwise we require evidence of a correct method for any marks to be awarded.**

### **Diagrams**

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

### **Work erased or crossed out**

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

### **Choice**

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

**AS/A-level Maths/Further Maths assessment objectives**

| AO         |        | Description   |
|------------|--------|---|
| <b>AO1</b> | AO1.1a | Select routine procedures   |
|            | AO1.1b | Correctly carry out routine procedures  |
|            | AO1.2  | Accurately recall facts, terminology and definitions                              |
| <b>AO2</b> | AO2.1  | Construct rigorous mathematical arguments (including proofs)                      |
|            | AO2.2a | Make deductions   |
|            | AO2.2b | Make inferences   |
|            | AO2.3  | Assess the validity of mathematical arguments                                     |
|            | AO2.4  | Explain their reasoning   |
|            | AO2.5  | Use mathematical language and notation correctly                                  |
| <b>AO3</b> | AO3.1a | Translate problems in mathematical contexts into mathematical processes           |
|            | AO3.1b | Translate problems in non-mathematical contexts into mathematical processes       |
|            | AO3.2a | Interpret solutions to problems in their original context                         |
|            | AO3.2b | Where appropriate, evaluate the accuracy and limitations of solutions to problems |
|            | AO3.3  | Translate situations in context into mathematical models                          |
|            | AO3.4  | Use mathematical models   |
|            | AO3.5a | Evaluate the outcomes of modelling in context                                     |
|            | AO3.5b | Recognise the limitations of models   |
|            | AO3.5c | Where appropriate, explain how to refine models                                   |

| Q            | Marking Instructions   | AO   | Marks    | Typical solution                               |
|--------------|------------------------|------|----------|--|
| 1            | Circles correct answer | 1.1b | B1       | $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ |
| <b>Total</b> |                        |      | <b>1</b> |  |

| Q            | Marking Instructions   | AO   | Marks    | Typical solution |
|--------------|------------------------|------|----------|------------------|
| 2            | Circles correct answer | 1.1b | B1       | $-120^\circ$     |
| <b>Total</b> |                        |      | <b>1</b> |                  |

| Q            | Marking Instructions | AO   | Marks    | Typical solution   |
|--------------|----------------------|------|----------|--|
| 3            | Ticks correct answer | 2.2a | B1       | $\mathbf{r} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ |
| <b>Total</b> |                      |      | <b>1</b> |  |

| Q            | Marking Instructions  | AO  | Marks    | Typical solution  |
|--------------|---|-----|----------|---|
| 4(a)         | Completes a rigorous argument to show the required result<br><br>Must begin with<br>$(r + 1)^2 - r^2 = \dots$ | 2.1 | R1       | $(r + 1)^2 - r^2 = r^2 + 2r + 1 - r^2$ $= 2r + 1 \text{ as required}$ |
| <b>Total</b> |   |     | <b>1</b> |   |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution   |
|--------------|---|------|----------|--|
| 4(b)         | Uses method of differences including at least the first two or last two terms   | 1.1a | M1       | $\sum_{r=1}^n (2r + 1) = \sum_{r=1}^n ((r + 1)^2 - r^2)$ $= \cancel{2^2} - 1^2$ $+ \cancel{3^2} - \cancel{2^2}$ $+ \dots$ $+ \dots$ $+ \cancel{n^2} - \cancel{(n-1)^2}$ $+ (n + 1)^2 - \cancel{n^2}$ $= (n + 1)^2 - 1$ $= n^2 + 2n + 1 - 1$ $= n^2 + 2n$ |
|              | Identifies and simplifies the two remaining terms   | 1.1a | M1       |  |
|              | Completes a rigorous argument to show the required result, including seeing at least the first two and the last two terms. Must begin with<br><br>$\sum_{r=1}^n (2r + 1) = \dots$ | 2.1  | R1       |  |
| <b>Total</b> |   |      | <b>3</b> |  |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution  |
|--------------|---|------|----------|---|
| 4(c)         | Recalls and states<br>$\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$  | 1.2  | B1       | $\sum_{r=1}^n r = \frac{1}{2}n(n + 1)$ $\therefore \sum_{r=1}^n (2r + 1) = 2 \times \frac{1}{2}n(n + 1) + n$ $= n^2 + n + n$ $= n^2 + 2n \text{ as required}$ |
|              | Splits the sum into two parts and uses their formula.   | 1.1a | M1       |   |
|              | Completes a clear argument to show the required result. Condone the lack of limits on the summation signs. Must begin with<br>$\sum_{r=1}^n (2r + 1) = \dots$ | 2.1  | R1       |   |
| <b>Total</b> |   |      | <b>3</b> |   |

|                       |  |  |          |  |
|-----------------------|--|--|----------|--|
| <b>Question total</b> |  |  | <b>7</b> |  |
|-----------------------|--|--|----------|--|

| Q | Marking Instructions   | AO   | Marks    | Typical solution   |
|---|--|------|----------|--|
| 5 | Expresses $w$ in terms of $z$<br>or<br>States the values of the sum, product and pairwise sum of roots of original equation.<br>Condone sign errors.   | 3.1a | M1       | $w = \frac{1}{2}z - 1$ so $z = 2w + 2$<br><br>Substituting in the original equation<br><br>$(2w + 2)^3 + 2(2w + 2)^2 - 5(2w + 2) - 3 = 0$<br>$8w^3 + 24w^2 + 24w + 8 + 8w^2 + 16w + 8$<br>$\quad\quad\quad - 10w - 10 - 3 = 0$<br>$8w^3 + 32w^2 + 30w + 3 = 0$ |
|   | Expresses $z$ correctly in terms of $w$<br>or<br>obtains correct value ( $-4$ ) of sum of roots of new equation  | 1.1b | A1       |  |
|   | Substitutes their expression for $z$ into original equation to form an equation in $w$<br>or<br>Expresses the pairwise sum of roots of new equation in terms $\sum \alpha$ and $\sum \alpha \beta$ | 1.1a | M1       |  |
|   | Simplifies their equation in $w$<br>or<br>Expresses product of roots of new equation in terms $\sum \alpha$ , $\sum \alpha \beta$ and $\alpha \beta \gamma$  | 1.1a | M1       |  |
|   | Obtains correct equation (any correct form)  | 1.1b | A1       |  |
|   | <b>Total</b>   |      | <b>5</b> |  |



| Q            | Marking Instructions                  | AO   | Marks    | Typical solution  |
|--------------|---------------------------------------|------|----------|---|
| 6(a)         | Obtains the correct equation of $E_2$ | 1.1b | B1       | Equation of $E_2$ is<br>$(x - 3)^2 + \frac{y^2}{4} = 1$ |
| <b>Total</b> |                                       |      | <b>1</b> |   |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution               |
|--------------|---|------|----------|--------------------------------|
| 6(b)         | Obtains a correct answer<br><br>Condone a correct sequence of transformations | 1.1b | B1       | Reflection in the line $y = x$ |
| <b>Total</b> |   |      | <b>1</b> |                                |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution |
|--------------|---|------|----------|------------------|
| 6(c)         | Two ellipses, one crossing the positive $x$ -axis and the other crossing the positive $y$ -axis | 1.1b | B1       |                  |
|              | Correct axis intercepts shown for both ellipses   | 1.1b | B1       |                  |
|              | At least one correct tangent drawn<br><br>Condone $x = 2$ or $y = 2$                            | 1.1b | B1       |                  |
|              | Both lines drawn and labelled correctly   | 2.2b | B1       |                  |
| <b>Total</b> |   |      | <b>4</b> |                  |

| Q            | Marking Instructions  | AO  | Marks    | Typical solution   |
|--------------|---|-----|----------|--|
| 6(d)         | Uses the fact that $E_3$ is a reflection of $E_2$ in the line $y = x$   | 2.4 | E1       | Points on $E_2$ and $E_3$ joined by $L_A$ are symmetrical about $y = x$ , therefore the line is perpendicular to $y = x$ and has a gradient of $-1$ and is of the form $y = -x + c$ or $x + y = c$ |
|              | Explains that the tangent is perpendicular to the line $y = x$ and concludes that its equation is $x + y = c$ | 2.4 | E1       |  |
| <b>Total</b> |   |     | <b>2</b> |  |

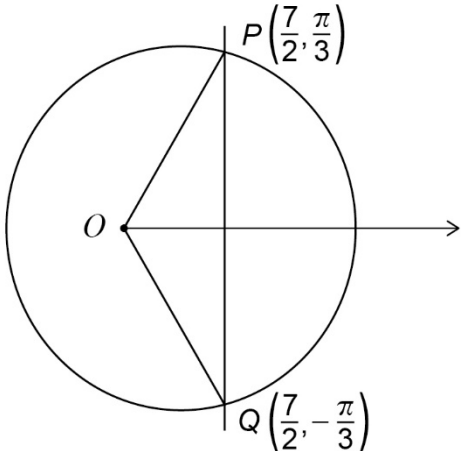
|                       |  |  |          |  |
|-----------------------|--|--|----------|--|
| <b>Question total</b> |  |  | <b>8</b> |  |
|-----------------------|--|--|----------|--|

| Q | Marking Instructions   | AO   | Marks | Typical Solution   |
|---|--|------|-------|--|
| 7 | Obtains derivatives of $x$ and $y$   | 1.1a | M1    | $\begin{aligned} \dot{x} &= -12 \cos^2 t \sin t \\ \dot{y} &= 12 \sin^2 t \cos t \\ \dot{x}^2 + \dot{y}^2 &= 144 \cos^4 t \sin^2 t \\ &\quad + 144 \sin^4 t \cos^2 t \\ &= 144 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) \\ &= 144 \cos^2 t \sin^2 t \\ \sqrt{\dot{x}^2 + \dot{y}^2} &= 12 \cos t \sin t \\ S &= 2\pi \int_0^{\frac{\pi}{2}} y \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= 2\pi \int_0^{\frac{\pi}{2}} 4 \sin^3 t (12 \cos t \sin t) dt \\ &= 96\pi \int_0^{\frac{\pi}{2}} \sin^4 t \cos t dt \\ &= 96\pi \left[ \frac{\sin^5 t}{5} \right]_0^{\frac{\pi}{2}} \\ &= \frac{96\pi}{5} \end{aligned}$ |
|   | Obtains correct expression for $\dot{x}^2 + \dot{y}^2$   | 1.1b | A1    |  |
|   | Uses trig identity to simplify their expression for $\dot{x}^2 + \dot{y}^2$  | 2.2a | B1    |  |
|   | Substitutes their expression for $\dot{x}^2 + \dot{y}^2$ into the formula for surface area<br>Condone missing limits of integration and missing “ $2\pi$ ” | 1.1a | M1    |  |
|   | Obtains correct expression for the surface area including correct limits of integration  | 1.1b | A1    |  |
|   | Obtains $k \sin^5 t$ by integration<br>Condone missing limits of integration   | 1.1b | A1    |  |
|   | Completes a rigorous argument to show the required result  | 2.1  | R1    |  |
|   | <b>Total</b>   |      |       |  |

| Q | Marking Instructions  | AO   | Marks | Typical solution   |
|---|---|------|-------|--|
| 8 | Expresses $z$ and $z^*$ in terms of $x$ and $y$ and substitutes these in the first equation                             | 1.1a | M1    | Let $z = x + iy$<br>Then<br>$ x - iy - 1 - 2i  =  x + iy - 3 $<br>$(x - 1)^2 + (y + 2)^2 = (x - 3)^2 + y^2$<br>$-2x + 1 + 4y + 4 = -6x + 9$<br>$y = 1 - x$ |
|   | Deduces a correct linear Cartesian equation for first equation  | 2.2a | A1    | and<br>$(x - a)^2 + y^2 = 9$   |
|   | Obtains correct Cartesian equation for second equation or states that this represents a circle centre $(a, 0)$ radius 3 | 1.1b | B1    | Solving simultaneously,<br>$(x - a)^2 + (1 - x)^2 = 9$<br>$2x^2 + (-2a - 2)x + (a^2 - 8) = 0$  |
|   | Solves their Cartesian equations simultaneously   | 1.1a | M1    | $\Delta \geq 0$ so<br>$(2a + 2)^2 - 4(2)(a^2 - 8) \geq 0$<br>$a^2 - 2a - 17 \leq 0$<br>$(a - 1)^2 \leq 18$   |
|   | States that the discriminant is non-negative or creates an inequality based on this                                     | 2.4  | E1    | So $1 - \sqrt{18} \leq a \leq 1 + \sqrt{18}$   |
|   | Completes a rigorous argument to obtain the correct range of values for $a$   | 2.1  | R1    | And $a$ must lie in the interval<br>$[1 - 3\sqrt{2}, 1 + 3\sqrt{2}]$   |
|   | <b>Total</b>  |      |       | <b>6</b>   |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution  |
|--------------|---|------|----------|---|
| 9(a)         | Obtains $x = \frac{7}{4}$ oe  | 1.1b | B1       | $r = \frac{7}{4} \sec \theta$ $r \cos \theta = \frac{7}{4}$ And $x = r \cos \theta$<br><br>So in Cartesian coordinates: $x = \frac{7}{4}$<br>which is perpendicular to the $x$ -axis<br><br>$\therefore L$ is perpendicular to the initial line |
|              | Explains that $L$ is perpendicular to the initial line or $x$ -axis | 2.1  | E1       |   |
| <b>Total</b> |   |      | <b>2</b> |   |

| Q            | Marking Instructions   | AO   | Marks    | Typical solution   |
|--------------|--|------|----------|--|
| 9(b)         | Obtains equation in $\theta$ or $r$                                  | 1.1a | M1       | At points of intersection<br>$\frac{7}{4} \sec \theta = 3 + \cos \theta$ $\cos^2 \theta + 3 \cos \theta - \frac{7}{4} = 0$ $4 \cos^2 \theta + 12 \cos \theta - 7 = 0$ $\cos \theta = -\frac{7}{2} \text{ (reject as } -1 \leq \cos \theta \leq 1)$ or $\cos \theta = \frac{1}{2}$<br><br>Points are $\left(\frac{7}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{7}{2}, -\frac{\pi}{3}\right)$ |
|              | Rearranges and solves for $\cos \theta$ or $\sec \theta$ or $r$      | 3.1a | M1       |  |
|              | Rejects, with a reason, the impossible value of $\cos \theta$ or $r$ | 2.2a | E1       |  |
|              | Obtains correct value of $\cos \theta$ or $r$                        | 1.1b | A1       |  |
|              | Obtains correct polar coordinates                                    | 1.1b | A1       |  |
| <b>Total</b> |  |      | <b>5</b> |  |

| Q                     | Marking Instructions  | AO   | Marks     | Typical solution   |
|-----------------------|---|------|-----------|--|
| 9(c)                  | Identifies the required region.                                     | 3.1a | M1        |  <p> <math>PQ = \frac{7\sqrt{3}}{2}</math> </p> <p>                     Area of triangle OPQ<br/> <math>= \frac{1}{2} \times \frac{7\sqrt{3}}{2} \times \frac{7}{4} = \frac{49\sqrt{3}}{16}</math> </p> <p>                     Area of sector OPQ = <math>2 \left( \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 d\theta \right)</math><br/>                     (by symmetry)<br/> <math>= \int_0^{\frac{\pi}{3}} (3 + \cos \theta)^2 d\theta</math><br/> <math>= \int_0^{\frac{\pi}{3}} (9 + 6 \cos \theta + \cos^2 \theta) d\theta</math><br/> <math>= \int_0^{\frac{\pi}{3}} (9 + 6 \cos \theta + \frac{1}{2} \cos 2\theta + \frac{1}{2}) d\theta</math><br/> <math>= \left[ \frac{19\theta}{2} + 6 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{\frac{\pi}{3}}</math><br/> <math>= \left( \frac{19\pi}{6} + 3\sqrt{3} + \frac{\sqrt{3}}{8} \right) - 0</math><br/> <math>= \frac{19\pi}{6} + \frac{25\sqrt{3}}{8}</math> </p> <p>                     Required area = <math>\frac{19\pi}{6} + \frac{25\sqrt{3}}{8} - \frac{49\sqrt{3}}{16}</math><br/> <math>= \frac{19\pi}{6} + \frac{\sqrt{3}}{16}</math> </p> |
|                       | Obtains correct area of triangle                                    | 1.1b | B1        |  |
|                       | Obtains $k \int (3 + \cos \theta)^2 d\theta$ with or without limits | 3.1a | M1        |  |
|                       | Uses $\cos^2 \theta = \frac{1}{2} \cos 2\theta + \frac{1}{2}$       | 3.1a | M1        |  |
|                       | Integrates their (at least three-part) expression correctly         | 1.1b | A1F       |  |
|                       | Obtains correct area of sector or half of sector                    | 1.1b | A1        |  |
|                       | Obtains exact correct answer  | 1.1b | A1        |  |
| <b>Total</b>          |   |      | <b>7</b>  |  |
| <b>Question total</b> |   |      | <b>14</b> |  |

| Q            | Marking Instructions     | AO   | Marks    | Typical solution   |
|--------------|--------------------------|------|----------|--|
| 10(a)        | Separates the variables  | 1.1a | M1       | $\frac{dy}{dt} = -ky$ $\int \frac{1}{y} dy = \int -k dt$ So $y = y_0 e^{-kt}$ $0.84y_0 = y_0 e^{-k}$ $k = 0.174$ |
|              | Deduces exponential form | 2.2a | M1       |  |
|              | Forms correct equation   | 3.3  | M1       |  |
|              | Obtains correct answer   | 1.1b | A1       |  |
| <b>Total</b> |                          |      | <b>4</b> |  |

| Q            | Marking Instructions  | AO  | Marks    | Typical solution                     |
|--------------|---|-----|----------|--------------------------------------|
| 10(b)        | Forms DE with three terms and $\frac{dy}{dt}$   | 3.3 | M1       | $\frac{dy}{dt} = -0.174y + 45 + 20t$ |
|              | Obtains correct DE (consistent with their answer to part (a)) may use $k$ in place of 0.174 | 3.3 | A1F      |                                      |
| <b>Total</b> |   |     | <b>2</b> |                                      |

| Q            | Marking Instructions  | AO   | Marks    | Typical solution   |
|--------------|---|------|----------|--|
| 10(c)        | Splits into CF and PI<br>Or<br>Finds an integrating factor  | 3.1a | M1       | $\frac{dy}{dt} + 0.174y = 45 + 20t$ CF: $y = Ae^{-0.174t}$<br>PI: $y = p + qt$ $\dot{y} = q$ $q + 0.174p + 0.174qt = 45 + 20t$ $q = \frac{20}{0.174} = 115$ $p = \frac{45 - q}{0.174} = -402$ General solution:<br>$y = Ae^{-0.174t} - 402 + 115t$ $t = 0 \Rightarrow y = 340$ So $A = 742$<br>$y = 742e^{-0.174t} - 402 + 115t$ |
|              | Obtains PI of the form $p + qt$<br>Or<br>Uses integration by parts to integrate the $te^{kt}$ term            | 1.1a | M1       |  |
|              | Obtains correct general solution from their value of $k$  | 1.1b | A1F      |  |
|              | Uses initial conditions to find the unknown constant for a general solution that includes an exponential term | 3.3  | M1       |  |
|              | Obtains correct solution  | 1.1b | A1       |  |
| <b>Total</b> |   |      | <b>5</b> |  |

| <b>Q</b>     | <b>Marking Instructions</b>                               | <b>AO</b> | <b>Marks</b> | <b>Typical solution</b>  |
|--------------|---|-----------|--------------|--|
| <b>10(d)</b> | One reasonable limitation of their model for limitation 1 | 3.5b      | E1           | Births would occur at a particular time of year, not at a steady rate                        |
|              | One reasonable limitation of their model for limitation 2 | 3.5b      | E1           | Over a long period of time the population would increase indefinitely according to the model |
|              | <b>Total</b>  |           | <b>2</b>     |  |

|  |                       |  |           |  |
|--|-----------------------|--|-----------|--|
|  | <b>Question total</b> |  | <b>13</b> |  |
|--|-----------------------|--|-----------|--|

| Q   | Marking Instructions   | AO   | Marks    | Typical solution  |
|---|--|------|----------|---|
| 11  | Obtains a position vector of a point on $L_1$ or $L_2$   | 2.5  | B1       | <p>The position vector of a point on <math>L_1</math> is</p> $\begin{bmatrix} -1 \\ 5 \\ -5/2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -2 \\ 3/2 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 5 \\ -5/2 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$ <p>The position vector of a point on <math>L_2</math> is</p> $\begin{bmatrix} 1/2 \\ 14 \\ -12 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ m \\ p \end{bmatrix}$ $\begin{bmatrix} 0.5 + \mu \\ 14 + m\mu \\ -12 + p\mu \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 1 & 2 \\ 1 & b & 4 \\ -3 & -2 & c \end{bmatrix} \begin{bmatrix} -1 + 6\lambda \\ 5 - 4\lambda \\ -2.5 + 3\lambda \end{bmatrix}$ $= \begin{bmatrix} 0.5 - 3\lambda + 5 - 4\lambda - 5 + 6\lambda \\ -1 + 6\lambda + 5b - 4b\lambda - 10 + 12\lambda \\ 3 - 18\lambda - 10 + 8\lambda - 2.5c + 3c\lambda \end{bmatrix}$ $= \begin{bmatrix} 0.5 - \lambda \\ 5b - 11 + \lambda(18 - 4b) \\ -7 - 2.5c + \lambda(3c - 10) \end{bmatrix}$ <p><math>0.5 + \mu = 0.5 - \lambda</math></p> <p>Therefore, when <math>\mu = 0</math>, <math>\lambda = 0</math> so we can equate the constant terms in the equation</p> $\left. \begin{array}{l} 14 = -11 + 5b \\ -12 = -7 - 5c/2 \end{array} \right\} \rightarrow \begin{array}{l} b = 5 \\ c = 2 \end{array}$ $\left. \begin{array}{l} \mu = -\lambda \\ m\mu = -2\lambda \\ p\mu = -4\lambda \end{array} \right\} \rightarrow \begin{array}{l} m = 2 \\ p = 4 \end{array}$ |
|   | Obtains a direction vector for $L_1$ or $L_2$ , ISW  | 1.1b | B1       |   |
|   | Obtains vector equations for both $L_1$ and $L_2$  | 2.5  | B1       |   |
|   | Forms a matrix equation equating the image of a general point on $L_1$ with a general point on $L_2$ | 3.1a | M1       |   |
|   | Condones same parameter used twice   |      |          |   |
|   | Collects and simplifies terms  | 1.1b | M1       |   |
|   | Compares their constant terms to obtain a value for at least one of $b$ or $c$                       | 3.1a | M1       |   |
|   | Must have used different parameters for their general points   |      |          |   |
| Obtains correct values of both $b$ and $c$                              | 1.1b   | A1   |          |   |
| Uses their $b$ and $c$ to obtain a value for at least one of $m$ or $p$ | 2.2a   | M1   |          |   |
| Obtains correct values of $m$ and $p$                                   | 1.1b   | A1   |          |   |
| <b>Total</b>  |  |      | <b>9</b> |   |



| Q     | Marking Instructions   | AO   | Marks | Typical Solution   |
|-------|--|------|-------|--|
| 12(a) | Selects a method to find the required result by integrating by parts   | 1.1a | M1    |  |
|       | Obtains the correct expressions for $u'$ and $v$ when integrating the first time                               | 1.1b | A1    |  |
|       | Correctly applies integration by parts formula the first time  | 1.1b | A1    | $S_n = \int_0^a x^n \sinh x \, dx$ $u = x^n \quad v' = \sinh x$ $u' = nx^{n-1} \quad v = \cosh x$  |
|       | Correctly applies integration by parts formula to an integral of the form $\int_0^a x^r \cosh x \, dx$         | 3.1a | M1    | $S_n = [x^n \cosh x]_0^a - \int_0^a nx^{n-1} \cosh x \, dx$ $= a^n \cosh a - n \int_0^a x^{n-1} \cosh x \, dx$   |
|       | Obtains correct result from 2 <sup>nd</sup> integration by parts and substitutes limits correctly in all terms | 1.1b | A1    | $S_n = a^n \cosh a - n \left( [x^{n-1} \sinh x]_0^a - \int_0^a (n-1)x^{n-2} \sinh x \, dx \right)$ $S_n = a^n \cosh a - n(a^{n-1} \sinh a - (n-1)S_{n-2})$ |
|       | Obtains an expression for $S_n$ in terms of $S_{n-2}$  | 1.1a | M1    | $S_n = a^n \cosh a - na^{n-1} \sinh a + n(n-1)S_{n-2}$ $S_n = n(n-1)S_{n-2} + a^n \cosh a - na^{n-1} \sinh a$  |
|       | Completes a rigorous argument to show the required result, including correct use of limits throughout          | 2.1  | R1    |  |
|       | <b>Total</b>   |      |       | <b>7</b>   |

| Q            | Marking Instructions   | AO   | Marks     | Typical Solution  |
|--------------|--|------|-----------|---|
| <b>12(b)</b> | Deduces that the integral is $S_4$ with $a=1$ and requires $S_2$ and $S_0$ | 2.2a | M1        | The required integral is $S_4$ with $a=1$<br>$S_0 = \int_0^1 \sinh x \, dx = [\cosh x]_0^1$ $= \cosh 1 - 1$ $S_2 = (2)(1)S_0 + \cosh 1 - 2 \sinh 1$ $= 2(\cosh 1 - 1) + \cosh 1 - 2 \sinh 1$ $= 3 \cosh 1 - 2 \sinh 1 - 2$ $S_4 = (4)(3)S_2 + \cosh 1 - 4 \sinh 1$ $= 12(3 \cosh 1 - 2 \sinh 1 - 2) + \cosh 1$ $- 4 \sinh 1$ $= 37 \cosh 1 - 28 \sinh 1 - 24$ $= \frac{37}{2}(e + e^{-1}) - 14(e - e^{-1}) - 24$ $= \frac{9}{2}e + \frac{65}{2}e^{-1} - 24$ |
|              | Uses the reduction formula once  | 3.1a | M1        |   |
|              | Uses the reduction formula a second time and finds $S_0$                   | 3.1a | M1        |   |
|              | Converts both hyperbolic expressions to exponentials                       | 1.1a | M1        |   |
|              | Completes a rigorous argument to show the required result                  | 2.1  | R1        |   |
|              | <b>Total</b>   |      | <b>5</b>  |   |
|              | <b>Question total</b>  |      | <b>12</b> |   |

| Q            | Marking Instructions                   | AO   | Marks    | Typical solution  |
|--------------|--|------|----------|---|
| 13(a)        | Obtains at least two correct solutions | 1.1a | M1       | $6\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$ $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ (in addition to given solutions) |
|              | Obtains all correct solutions          | 1.1b | A1       |   |
| <b>Total</b> |  |      | <b>2</b> |   |

| Q            | Marking Instructions   | AO   | Marks    | Typical solution  |
|--------------|--|------|----------|---|
| 13(b)        | Expands $(\cos \theta + i \sin \theta)^6$  | 1.1a | M1       | By de Moivre's theorem<br>$\cos 6\theta + i \sin 6\theta = (\cos \theta + i \sin \theta)^6$ $= \cos^6 \theta + 6 \cos^5 \theta (i \sin \theta) + 15 \cos^4 \theta (i \sin \theta)^2 + 20 \cos^3 \theta (i \sin \theta)^3 + 15 \cos^2 \theta (i \sin \theta)^4 + 6 \cos \theta (i \sin \theta)^5 + (i \sin \theta)^6$ Equating real parts<br>$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$ $= \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$ $= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta - 30 \cos^4 \theta + 15 \cos^6 \theta - 1 + 3 \cos^2 \theta - 3 \cos^4 \theta + \cos^6 \theta$ $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$ |
|              | Equates real parts   | 3.1a | M1       |   |
|              | Obtains correct expression for real part in terms of powers of $\cos \theta$ and $\sin \theta$ | 1.1b | A1       |   |
|              | Uses trig identity to express real part in terms of $\cos \theta$                              | 3.1a | M1       |   |
|              | Completes a rigorous argument to obtain the required result                                    | 2.1  | R1       |   |
| <b>Total</b> |  |      | <b>5</b> |   |

| Q     | Marking Instructions  | AO   | Marks    | Typical solution  |
|-------|---|------|----------|---|
| 13(c) | Uses the fact that either $\theta = \frac{\pi}{4}$ or $\theta = \frac{3\pi}{4}$ is a solution to the first equation to deduce that it is also a solution to the second equation | 2.2a | M1       | $\theta = \frac{\pi}{4} \Rightarrow \cos 6\theta = 0$ $\Rightarrow 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1 = 0$ from part (b)<br>$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \therefore \left( \cos \theta - \frac{1}{\sqrt{2}} \right)$ is a factor of<br>$32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$ |
|       | Uses the factor theorem   | 3.1a | M1       | Similarly $\cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$ and $\left( \cos \theta + \frac{1}{\sqrt{2}} \right)$ is also a factor of the expression.   |
|       | Multiplies their linear factors together  | 1.1a | M1       | So $\left( \cos \theta - \frac{1}{\sqrt{2}} \right) \left( \cos \theta + \frac{1}{\sqrt{2}} \right) = \left( \cos^2 \theta - \frac{1}{2} \right)$ is a factor and $(2 \cos^2 \theta - 1)$ is a factor   |
|       | Obtains the correct result (oe)   | 2.1  | R1       |   |
|       | <b>Total</b>  |      | <b>4</b> |   |

| Q     | Marking Instructions   | AO   | Marks      | Typical solution   |
|-------|--|------|------------|--|
| 13(d) | Divides the polynomial by their quadratic factor   | 3.1a | M1         | Let $c = \cos \theta$  |
|       | Solves their quartic equation as a quadratic in $c^2$  | 1.1a | M1         | $32c^6 - 48c^4 + 18c^2 - 1 = (2c^2 - 1)(16c^4 - 16c^2 + 1)$ For $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ , $2c^2 - 1 = 0$<br>So for the other four roots, which are the cosines of the other four solutions to $\cos 6\theta = 0$ ,      |
|       | Explains that the roots of the quartic correspond to the cosines of the angles found in part (a)   | 2.4  | E1         | $16c^4 - 16c^2 + 1 = 0$ Solving this as a quadratic in $c^2$ gives $c^2 = \frac{2 \pm \sqrt{3}}{4}$  |
|       | Obtains all correct roots of the quartic   | 1.1b | A1         | So $c = \pm \sqrt{\frac{2 \pm \sqrt{3}}{4}}$   |
|       | Uses a rigorous argument to obtain the required result, including a reason why that particular root corresponds to $\cos\left(\frac{11\pi}{12}\right)$ | 2.1  | R1         | Of the angles $\frac{\pi}{12}$ , $\frac{5\pi}{12}$ , $\frac{7\pi}{12}$ , and $\frac{11\pi}{12}$ , $\frac{11\pi}{12}$ has the negative cosine of the greatest magnitude<br>Therefore $\cos\left(\frac{11\pi}{12}\right) = -\sqrt{\frac{2+\sqrt{3}}{4}}$ |
|       | <b>Total</b>   |      | <b>5</b>   |  |
|       | <b>Question total</b>  |      | <b>16</b>  |  |
|       | <b>Paper total</b>   |      | <b>100</b> |  |