AQA

A-level
FURTHER MATHEMATICS
7367/1

Paper 1

Mark scheme

June 2023

Version: 1.0 Final

2 3 6 A 7 3 6 7 / 1 / M S

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
‘their’	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)
ISW	Ignore Subsequent Workings

Examiners should consistently apply the following general marking principles:

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking instructions	AO	Marks	Typical solution
1	Circles correct answer	1.1b	B1	0
	Total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
2	Circles correct answer	1.2	B1	$\lvert z-2+3\mathrm{i}\rvert = 2$
	Total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
3	Circles correct answer	2.2a	B1	(3, 0)
	Total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
4	Ticks correct answer	1.2	B1	$\mathrm{f}(t) = 7\mathrm{e}^{-t} + 2\mathrm{e}^{-2t}$
	Total		**1**	

Q	Marking instructions	AO	Marks	Typical solution
5(a)	Obtains a correct expression for $\mathrm{f}(r+1)$ and forms an expression for the difference $\mathrm{f}(r+1)-\mathrm{f}(r)$ Eg $2^{r+1}(r-1)-2^{r}(r-2)$	1.1a	M1	$\mathrm{f}(r+1)-\mathrm{f}(r)=2^{r+1}(r-1)-2^{r}(r-2)$ $=2^{r}(2(r-1)-(r-2))$ $=2^{r}(2r-2-r+2)$ $=r2^{r}$
	Completes a rigorous argument to obtain the required result Must include $\mathrm{f}(r+1)-\mathrm{f}(r)$, $r2^{r}$ and at least one intermediate step between $2^{r+1}(r-1)-2^{r}(r-2)$ and $r2^{r}$	2.1	R1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
5(b)	Writes at least two lines of subtracting terms, using part (a) Allow $\mathrm{f}(2)-\mathrm{f}(1)$ etc	1.1a	M1	$\sum_{r=1}^{n} r2^{r}=\sum_{r=1}^{n}\mathrm{f}(r+1)-\mathrm{f}(r)$ $=0-(-2)$ $+8\quad-0$ $+32\quad-8$ $+\ldots$ $+2^{n}(n-2)-2^{n-1}(n-3)$ $+2^{n+1}(n-1)-2^{n}(n-2)$ $\sum_{r=1}^{n} r2^{r}=2^{n+1}(n-1)-2(-1)$ $=2^{n+1}(n-1)+2$
	Writes at least two consecutive lines showing cancellation (PI)	2.5	M1	
	Correctly reduces the expression to two terms	1.1b	A1	
	Completes a reasoned argument using the method of differences to reach the required result. This mark is only available if at least the first two lines and the last two lines, to reach the required result, are seen.	2.1	R1	
	Subtotal		**4**	

	Question total		**6**	

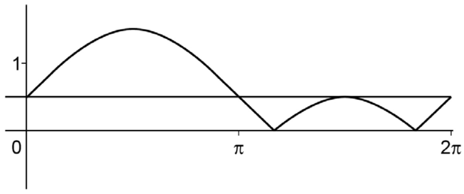
Q	Marking instructions	AO	Marks	Typical solution
6(a)(i)	Uses an appropriate method to obtain the value of λ_1	1.1a	M1	$\frac{1}{10}\begin{bmatrix} a & a & -6 \\ 0 & 10 & 0 \\ 9 & 14 & -13 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} a-18 \\ 0 \\ -30 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ $\lambda_1 = -1$
	Obtains the correct value of λ_1	1.1b	A1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
6(a)(ii)	Uses eigenvector definition to set up a linear equation in a	1.1a	M1	$10 = 18 - a$ $a = 8$
	Deduces the correct value of a	2.2a	A1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
6(b)	Uses an appropriate method to find λ or c PI by $\lambda_3 = 0.5$	3.1a	M1	$\frac{1}{10}\begin{bmatrix} 8 & 8 & -6 \\ 0 & 10 & 0 \\ 9 & 14 & -13 \end{bmatrix}\begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8c-6 \\ 0 \\ 9c-13 \end{bmatrix} = \lambda_3 \begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix}$ $\frac{1}{10}(8c-6) = \lambda_3 c$ and $\frac{1}{10}(9c-13) = \lambda_3$ $\frac{1}{10}(8c-6) = \frac{c}{10}(9c-13)$ $9c^2 - 21c + 6 = 0$ $c = 2$ and $c = \frac{1}{3}$ (reject as $c \in \mathbb{Z}$) $c = 2$ and $\lambda_3 = 0.5$
	Forms an equation in λ or c PI by $\lambda_3 = 0.5$	1.1a	M1	
	Finds the correct λ_3	1.1b	A1	
	Deduces the correct c NMS = 0/4 if λ_3 wrong	2.2a	A1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
6(c)	Deduces a correct **U**, ft their c only. Condone "c"	2.2a	B1F	$\mathbf{U}=\begin{bmatrix}1&1&2\\0&1&0\\3&1&1\end{bmatrix}$ $\mathbf{D}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&0.5\end{bmatrix}$ $\mathbf{U}^{-1}=\frac{1}{5}\begin{bmatrix}-1&-1&2\\0&5&0\\3&-2&-1\end{bmatrix}$
	Deduces the value of **D**, ft their λ_1 and λ_3. Must be compatible with their **U**. Condone "λ_1" and "λ_3"	2.2a	B1F	
	Obtains $\mathbf{U}^{-1}$, ft their **U**	1.1b	B1F	
	Subtotal		**3**	

	Question total		**11**	

Q	Marking instructions	AO	Marks	Typical solution
7	Sketches graph of $y = \sin x + \frac{1}{2}$ PI by graph of $y = \left\|\sin x + \frac{1}{2}\right\|$ or considers one equation or inequality without modulus sign eg $\sin x + \frac{1}{2} = \frac{1}{2}$ or $\left(\sin x + \frac{1}{2}\right)^2 = \frac{1}{4}$	3.1a	M1	 $\{x : 0 \le x \le \pi\} \cup \left\{\frac{3\pi}{2}\right\} \cup \{2\pi\}$
	Obtains the set of values $0 \le x \le \pi$ Condone $0 < x < \pi$	2.2a	A1	
	Obtains a graph of $y = \left\|\sin x + \frac{1}{2}\right\|$ with the correct shape or Obtains the other equation or inequality without modulus sign eg $\sin x + \frac{1}{2} = -\frac{1}{2}$ or Obtains two critical values from a quadratic in $\sin x$	1.1a	M1	
	Obtains $3\pi/2$	1.1b	A1	
	Obtains a completely correct answer, and expresses it using set notation. eg $[0, \pi] \cup \left\{\frac{3\pi}{2}, 2\pi\right\}$ $\{x : 0 \le x \le \pi\} \cup \left\{x : x = \frac{3\pi}{2}\right\} \cup \{x : x = 2\pi\}$ Condone $\left\{x : 0 \le x \le \pi, \frac{3\pi}{2}, 2\pi\right\}$	2.5	A1	
	Total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
8(a)	Deduces x-coordinates of stationary points	2.2a	B1	$\frac{\pi}{2}$ and $\frac{3\pi}{2}$
	Subtotal		**1**	

Q	Marking instructions	AO	Marks	Typical solution
8(b)	Obtains exactly three values of y for correct values of x , can be unsimplified.	1.1a	M1	<table><tr><td>x</td><td>0</td><td>$\pi/2$</td><td>π</td></tr><tr><td>y</td><td>1</td><td>e</td><td>1</td></tr></table> Estimate $=\frac{1}{3}\times\frac{\pi}{2}(1+1+4e)$ = 6.74
	Uses Simpson’s rule correctly Condone 5 ordinates.	1.1a	M1	
	Obtains correct value AWRT 6.74	1.1b	A1	
	Subtotal		**3**	

Q	Marking instructions	AO	Marks	Typical solution
8(c)	Infers that a larger number of ordinates/strips could give a more accurate result	2.2b	E1	By using Simpson’s rule with 5 ordinates
	Subtotal		**1**	

	Question total		**5**	

Q	Marking instructions	AO	Marks	Typical solution
9(a)	Obtains two vectors in the plane of the triangle	1.1b	B1	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} -3 \\ -9 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 18 \\ -6 \\ -9 \end{bmatrix}$ $\text{Area} = \frac{1}{2}\left\lvert \overrightarrow{AB} \times \overrightarrow{AC} \right\rvert = \frac{1}{2}\sqrt{18^2 + (-6)^2 + (-9)^2}$ $= \frac{21}{2}$
	Selects a method to find the area of a triangle for example by taking the vector product of their two vectors	3.1a	M1	
	Uses a correct formula for the area of a triangle	1.2	M1	
	Obtains the correct area with no incorrect working	1.1b	A1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
9(b)	Forms the scalar product of their normal and a position vector of a point in Π	1.1a	M1	$d = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix} = 4$ $\mathbf{r} \bullet \begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix} = 4$
	Obtains a correct equation of Π	1.1b	A1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
9(c)	Selects a method to find the parallel plane that *P* lies in by using the scalar product of their normal vector and *P* to obtain their "–8" or Uses a correct formula to find the required distance or Substitutes equation of perpendicular line from P to Π into equation of Π	3.1a	M1	$D_2 = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix} = -8$ $\lvert\mathbf{n}\rvert = \sqrt{6^2 + (-2)^2 + (-3)^2} = 7$ $d_1 = \frac{4}{7}$ & $d_2 = \frac{-8}{7}$ Distance of P from $\Pi = \frac{4}{7} - \frac{-8}{7} = \frac{12}{7}$
	Uses the magnitude of their normal vector to divide their "4" and "–8" in order to find the distance of *P* from Π or Substitutes correctly into a formula for the required distance	2.2a	M1	
	Obtains the correct distance of *P* from Π	1.1b	A1	
	Subtotal		**3**	

	Question total		**9**	

Q	Marking instructions	AO	Marks	Typical solution
10(a)	Correctly obtains the three components of the image of a general point under $\mathbf{M}$ in terms of c or using $c=3$	1.1b	B1	$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & -1 & -2 \\ 1 & 2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-y+z \\ -x-y-2z \\ x+2y+cz \end{bmatrix}$
	Substitutes their components into $x+5y+3z$	1.1a	M1	$x'+5y'+3z' = (2x-y+z)+5(-x-y-2z)+3(x+2y+cz)$ $= (2-5+3)x+(-1-5+6)y+(1-10+3c)z$ $= (-9+3c)z$
	Completes a reasoned argument to show that every image point lies in the plane $x+5y+3z=0$ when $c=3$	2.1	R1	So every image point lies in the plane $x+5y+3z=0$ when $c=3$
	Subtotal		**3**	

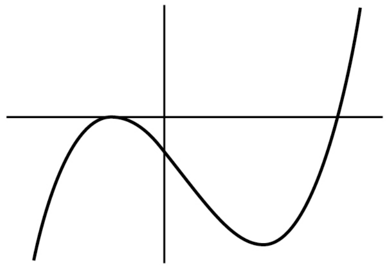
Q	Marking instructions	AO	Marks	Typical solution
10(b)(i)	Obtains an expression for $\lvert\mathbf{M}\rvert$ Can be seen in **(a)**	1.1a	M1	$\lvert\mathbf{M}\rvert = 2(-c+4)+1(-c+2)+1(-2+1)$ $= -3c+9$ $\therefore c \neq 3$
	Deduces correct restriction on c	2.2a	A1	
	Subtotal		**2**	

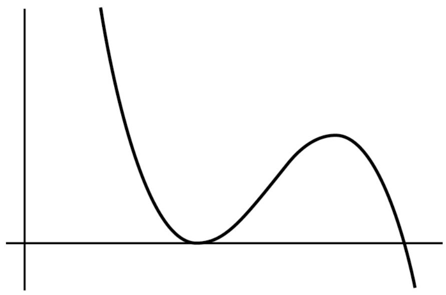
Q	Marking instructions	AO	Marks	Typical solution
10(b)(ii)	Obtains matrix of minors/cofactors with at least **four** correct elements PI by transposed form. Condone overall sign error on each element.	1.1a	M1	minors $\begin{bmatrix} -c+4 & -c+2 & -1 \\ -c-2 & 2c-1 & 5 \\ 3 & -3 & -3 \end{bmatrix}$ Cofactors $\begin{bmatrix} -c+4 & c-2 & -1 \\ c+2 & 2c-1 & -5 \\ 3 & 3 & -3 \end{bmatrix}$ $\mathbf{M}^{-1} = \dfrac{1}{-3c+9} \begin{bmatrix} -c+4 & c+2 & 3 \\ c-2 & 2c-1 & 3 \\ -1 & -5 & -3 \end{bmatrix}$
	Obtains matrix of minors/cofactors with at least **seven** correct elements PI by transposed form. Condone overall sign error on each element.	1.1a	M1	
	Obtains **correct** matrix of minors/cofactors PI by transposed form. Condone overall sign error on each element.	1.1b	A1	
	Deduces fully correct, simplified answer	2.2a	A1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
10(b)(iii)	Obtains their correct $\mathbf{M}^{-1}$ using $c=4$. Need not be simplified.	3.1a	B1F	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 0 & 6 & 3 \\ 2 & 7 & 3 \\ -1 & -5 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -6 \\ 13 \end{bmatrix}$
	Forms their product $\mathbf{M}^{-1}\begin{bmatrix} -3 \\ -6 \\ 13 \end{bmatrix}$ Condone $\mathbf{M}^{-1}$ in terms of c	1.1a	M1	$= \frac{1}{-3}\begin{bmatrix} 3 \\ -9 \\ -6 \end{bmatrix}$
	Completes a reasoned argument to obtain the correct solution. Do not accept $\mathbf{r} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ But do accept $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$ NMS or any method **NOT** using $\mathbf{M}^{-1}\mathbf{v}$ scores 0 marks	2.1	R1	$x=-1,\ y=3,\ z=2$
	Subtotal		**3**	

	Question total		**12**	

Q	Marking instructions	AO	Marks	Typical solution
11(a)(i)	Obtains one factor of $\mathrm{f}(x)$	1.1a	M1	$\mathrm{f}(x)=(x-5)(2x+3)^2$
	Deduces the correct factorisation	2.2a	A1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
11(a)(ii)	Deduces that $x<5$	2.2a	M1	
	Obtains a completely correct solution to the inequality	1.1b	A1	If $\mathrm{f}(x)<0$, $x<-\frac{3}{2}, -\frac{3}{2}<x<5$
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
11(b)	Obtains ±5.5 or ±12 ft 7 + their critical values from **(a)(ii)**	2.2a	M1	
	Obtains 5.5 and 12 ft 7 + their critical values from **(a)(ii)**	2.2a	A1F	
	Obtains a completely correct solution to the inequality	1.1b	A1	$\mathrm{g}(x)=0 \Rightarrow x=\frac{11}{2}, x=12$ If $\mathrm{g}(x)\le 0$, $x=\frac{11}{2}$ or $x\ge 12$
	Subtotal		**3**	

	Question total		**7**	

Q	Marking instructions	AO	Marks	Typical solution
12(a)	Uses hyperbolic identities correctly to express $\tanh 2x$ as a quotient	1.1a	M1	$\tanh 2x = \dfrac{\sinh 2x}{\cosh 2x} = \dfrac{2\sinh x\cosh x}{\cosh^2 x + \sinh^2 x}$ $= \dfrac{\dfrac{2\sinh x\cosh x}{\cosh^2 x}}{\dfrac{\cosh^2 x + \sinh^2 x}{\cosh^2 x}} = \dfrac{2\tanh x}{1+\tanh^2 x}$
	Completes a reasoned argument to obtain the required result	2.1	R1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
12(b)(i)	Correctly states range ACF	1.2	B1	$0 < \mathrm{f}(x) < 1$
	Subtotal		**1**	

Q	Marking instructions	AO	Marks	Typical solution
12(b)(ii)	Uses range of f to show that $1+\tanh^2 x < 2$	3.1a	M1	For $x > 0$, $0 < \tanh x < 1 \Rightarrow 1 < 1+\tanh^2 x < 2$ $\therefore \dfrac{2}{1+\tanh^2 x} > 1$ and $\dfrac{2\tanh x}{1+\tanh^2 x} > \tanh x$ $\therefore \tanh 2x > \tanh x$
	Deduces that $\dfrac{2}{1+\tanh^2 x} > 1$	2.2a	M1	
	Completes a reasoned argument to obtain the required result Working backwards from the result to the range of $\tanh x$ scores 0/3	2.1	R1	
	Subtotal		**3**	

	Question total		**6**	

<table>
<tr><th>Q</th><th>Marking instructions</th><th>AO</th><th>Marks</th><th>Typical solution</th></tr>
<tr><td rowspan="5">13</td><td>Explains that numerator and denominator are both zero at $x = \pi$</td><td>2.4</td><td>E1</td><td rowspan="5">Let
$f(x) = x\sin 2x$
$g(x) = \cos\left(\frac{x}{2}\right)$
Then
$f(\pi) = 0$ and $g(\pi) = 0$
So, by l'Hôpital's rule,
$\lim_{x\to\pi}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\pi}\left(\frac{f'(x)}{g'(x)}\right)$
$f'(x) = 2x\cos 2x + \sin 2x$
$g'(x) = -\frac{1}{2}\sin\left(\frac{x}{2}\right)$
$f'(\pi) = 2\pi\cos 2\pi + \sin 2\pi = 2\pi$
$g'(\pi) = -\frac{1}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{1}{2}$
$\lim_{x\to\pi}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to\pi}\left(\frac{f'(x)}{g'(x)}\right) = \frac{2\pi}{-\frac{1}{2}} = -4\pi$</td></tr>
<tr><td>Obtains derivatives of numerator and denominator</td><td>1.1a</td><td>M1</td></tr>
<tr><td>Correctly evaluates both correct derivatives of numerator and denominator at $x = \pi$, may be unsimplified</td><td>1.1b</td><td>A1</td></tr>
<tr><td>Forms their $\frac{f'(\pi)}{g'(\pi)}$</td><td>1.1a</td><td>M1</td></tr>
<tr><td>Completes a reasoned argument using l'Hôpital's rule to obtain the required result, including a clear demonstration of the limiting process.
Can score E0 M1A1M1R1</td><td>2.1</td><td>R1</td></tr>
<tr><td></td><td>Total</td><td></td><td>5</td><td></td></tr>
</table>

Q	Marking instructions	AO	Marks	Typical solution
14(a)	Uses the range of $\cos\theta$ to obtain max and min values of r or obtains $k = 2$	2.4	M1	r is minimum when $\cos\theta = 1$, and maximum when $\cos\theta = -1$ So $\frac{1}{2} \le r \le 2$
	Completes a reasoned argument to obtain the correct inequality	2.1	R1	
	Subtotal		**2**	

Q	Marking instructions	AO	Marks	Typical solution
14(b)	Uses $x = r\cos\theta$	3.1a	M1	$5r + 3r\cos\theta = 4$ $5r = 4 - 3r\cos\theta$ $5\sqrt{x^2 + y^2} = 4 - 3x$ $25\left(x^2 + y^2\right) = (4 - 3x)^2$ $25y^2 = 16 - 24x - 16x^2$ $y^2 = \frac{1}{25}\left(16 - 24x - 16x^2\right)$
	Uses $r^2 = x^2 + y^2$ or $r = \sqrt{x^2 + y^2}$	3.1a	M1	
	Rearranges to make ky^2 the subject, having correctly removed any square root in their equation. Equation must be in terms of x and y only.	1.1a	M1	
	Obtains a correct result in required form ISW	1.1b	A1	
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
14(c)	Completes the square to bring together all the x terms	3.1a	M1	$y^2 = -\frac{16}{25}\left(x^2 + \frac{3}{2}x - 1\right)$ $= -\frac{16}{25}\left\{\left(x + \frac{3}{4}\right)^2 - \frac{25}{16}\right\}$ $= -\frac{16}{25}\left(x + \frac{3}{4}\right)^2 + 1$ $y^2 + \frac{16}{25}\left(x + \frac{3}{4}\right)^2 = 1$ Starting from $y^2 + \frac{16x^2}{25} = 1$: The required transformation is a translation by $\begin{bmatrix} -\frac{3}{4} \\ 0 \end{bmatrix}$
	Rearranges equation to obtain a form that enables them to deduce a transformation	1.1a	M1	
	Deduces that a horizontal translation is required. PI a vector $\begin{bmatrix} p \\ 0 \end{bmatrix}$	2.2a	M1	
	Deduces a translation by $\begin{bmatrix} -\frac{3}{4} \\ 0 \end{bmatrix}$ as the only transformation	2.2a	A1	
	Subtotal		**4**	

	Question total		**10**	

Q	Marking instructions	AO	Marks	Typical solution
15	Uses auxiliary equation	3.1a	M1	aux. equation $m^2 - 3m - 4 = 0 \Rightarrow m = 4, -1$ CF: $y = Ae^{-x} + Be^{4x}$
	Obtains correct complementary function	1.1b	A1	PI: $y = p\cos 2x + q\sin 2x + rx + s$ $y' = -2p\sin 2x + 2q\cos 2x + r$ $y'' = -4p\cos 2x - 4q\sin 2x$
	Uses correct form of particular integral	3.1a	M1	$-4p\cos 2x - 4q\sin 2x + 6p\sin 2x - 6q\cos 2x - 3r$ $-4p\cos 2x - 4q\sin 2x - 4rx - 4s = \cos 2x + 5x$
	Obtains first and second derivatives of their particular integral	1.1a	M1	$6p - 8q = 0$ $-6q - 8p = 1$ $q = -\frac{3}{50}, p = -\frac{2}{25}$
	Substitutes first and second derivatives of particular integral into DE	3.1a	M1	$-4r = 5 \Rightarrow r = -\frac{5}{4}$ $-3r - 4s = 0 \Rightarrow s = \frac{15}{16}$ $y = Ae^{-x} + Be^{4x} - \frac{2}{25}\cos 2x - \frac{3}{50}\sin 2x - \frac{5}{4}x + \frac{15}{16}$
	Forms a pair of simultaneous equations by comparing coefficients of $\cos 2x$ and $\sin 2x$ and solves them	3.1a	M1	
	Forms a pair of simultaneous equations by comparing coefficient of x and constant term and solves them	3.1a	M1	
	Obtains correct trig constants or linear constants	1.1b	A1	
	Completes a reasoned argument to obtain completely correct solution, including $y =$	2.1	R1	
	Total		**9**	

Q	Marking instructions	AO	Marks	Typical solution
16(a)	Selects a suitable method to find the required result by integration by parts or making an appropriate substitution/inspection	3.1a	M1	$u = \ln t \quad \frac{dv}{dt} = \frac{1}{t}$ $\frac{du}{dt} = \frac{1}{t} \quad v = \ln t$
	Applies their correct integration method to obtain $2\int \frac{1}{t}\ln t \, dt = \left[(\ln t)^2\right]$ or $\int \frac{1}{t}\ln t dt = \left[\frac{u^2}{2}\right]$ OE	1.1b	A1	$\int_{0.5}^{4} \frac{1}{t}\ln t \, dt = \left[(\ln t)^2\right]_{0.5}^{4} - \int_{0.5}^{4} \frac{1}{t}\ln t \, dt$ $\int_{0.5}^{4} \frac{1}{t}\ln t \, dt = \frac{1}{2}\left[(\ln t)^2\right]_{0.5}^{4}$ $= \frac{1}{2}\left[(\ln 4)^2 - (\ln 0.5)^2\right]$ $= \frac{1}{2}\left[4(\ln 2)^2 - (-\ln 2)^2\right]$ $\int_{0.5}^{4} \frac{1}{t}\ln t \, dt = \frac{3}{2}(\ln 2)^2$
	Substitutes limits and uses the laws of logs correctly to simplify their result of integration in terms of $\ln 2$	1.1a	M1	or let $u = \ln t$ $...du = ...\frac{1}{t}dt$ $\int_{0.5}^{4} \frac{1}{t}\ln t dt = \int_{\ln 0.5}^{\ln 4} u du$
	Completes a rigorous argument to show the required result. NMS = 0/4	2.1	R1	$= \left[\frac{u^2}{2}\right]_{\ln 0.5}^{\ln 4}$ $= \frac{(\ln 4)^2}{2} - \frac{(\ln 0.5)^2}{2}$ $= \frac{(2\ln 2)^2}{2} - \frac{(-\ln 2)^2}{2}$ $= \frac{3}{2}(\ln 2)^2$
	Subtotal		**4**	

Q	Marking instructions	AO	Marks	Typical solution
16(b)	Correctly obtains $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$ in any form.	1.1b	B1	$x=2t \quad y=\frac{1}{2}t^2-\ln t$ $\frac{dx}{dt}=2 \quad \frac{dy}{dt}=t-\frac{1}{t}$
	Substitutes y and their $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2$ into the integrand of the formula for Surface Area of Revolution.	1.2	M1	$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=t^2+2+\frac{1}{t^2}=\left(t+\frac{1}{t}\right)^2$ $S=2\pi\int y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $=2\pi\int_{0.5}^{4}\left(\frac{1}{2}t^2-\ln t\right)\sqrt{\left(t+\frac{1}{t}\right)^2}\,dt$
	Obtains correctly expanded integrand	1.1b	A1	$S=2\pi\int_{0.5}^{4}\left(\frac{1}{2}t^3+\frac{1}{2}t-t\ln t-\frac{1}{t}\ln t\right)dt$
	Selects integration by parts to calculate an integral of form $k\int t\ln t\,dt$	3.1a	M1	$u=\ln t \quad \frac{dv}{dt}=t$ $\frac{du}{dt}=\frac{1}{t} \quad v=\frac{1}{2}t^2$ $\int t\ln t\,dt=\frac{1}{2}t^2\ln t-\int\frac{1}{2}t\,dt=\frac{1}{2}t^2\ln t-\frac{1}{4}t^2$
	Obtains correct result of integration by parts for their $k\int t\ln t\,dt$. No limits needed at this stage.	1.1b	A1	$S=2\pi\left\{\left[\frac{1}{8}t^4+\frac{1}{4}t^2-\frac{1}{2}t^2\ln t+\frac{1}{4}t^2\right]_{0.5}^{4}-\frac{3}{2}(\ln 2)^2\right\}$ $S=2\pi\left\{32+4-\frac{1}{128}-\frac{1}{16}+4-\frac{1}{16}-8\ln 4+\frac{1}{8}\ln\left(\frac{1}{2}\right)-\frac{3}{2}(\ln 2)^2\right\}$ $S=\pi\left\{\frac{5103}{64}-\frac{129}{4}\ln 2-3(\ln 2)^2\right\}$
	Substitutes limits into their expression of the form $at^4+bt^2+ct^2\ln t$ and use of their answer to part (a).	1.1a	M1	
	Obtains $\pi\left\{\frac{5103}{64}-\frac{129}{4}\ln 2-3(\ln 2)^2\right\}$	2.1	R1	
	Subtotal		**7**	

	Question total		**11**	

	Paper total		**100**	